IV. Physics

SPACE SCIENCES DIVISION

A. Response of Supersonic Laminar Boundary Layer to a Moving External Pressure Field,

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1. Introduction

In a supersonic wind tunnel the laminar boundary layer on a model in the test section is irradiated by sound waves generated by the turbulent boundary layers on the tunnel side walls. It has long been thought that the effect of these sound waves is to cause the early transition from laminar to turbulent flow that is observed in wind tunnels. It is probably not feasible to calculate the response of a laminar boundary layer to a sound field as complicated as that existing in a wind tunnel, but a simpler, although closely related, problem can be solved by making use of the methods developed for linear stability theory. This problem was formulated many years ago by Lees and Lin (Ref. 1), but has never been solved. The boundary layer is replaced by a parallel flow having the velocity and temperature profiles of the true boundary layer at some particular location, and the radiating turbulent boundary layer on the tunnel ceiling is replaced by a wavy wall moving in the direction of the flow with velocity c_r^*, or dimensionless velocity $c_r = c_r^*/U_1^*$, where U is the mean velocity and the subscript 1 refers to the free stream. The asterisks refer to dimensional quantities. When the waviness is of small amplitude and the relative free-stream Mach number $M_{R1} = M_1(1 - c_r)$ is supersonic (M_1 = free-stream Mach number), a disturbance field is produced which propagates along the family of Mach waves which extend in the downstream direction. It is the interaction of this moving pressure field with the specified parallel flow which is to be studied.

2. Method of Solution

The laminar stability theory has been extensively studied over the last few years[1] by means of two computer programs, one for the inviscid theory and the other for the viscous theory. Both of these programs make use of analytic solutions of the stability equations in the free stream to provide the initial conditions for the numerical integration, which extends from the edge of the boundary layer to the wall. In the inviscid theory there are two solutions, and when $M_{R1} > 1$, one solution represents a family of incoming Mach waves, and the second solution

[1]Mack, L. M., *Boundary Layer Stability Theory*, Rev. 1, 1969 (JPL internal document).

a family of outgoing Mach waves. When $M_{R1} < 1$, the outgoing, or reflected, wave family (for neutral and amplified disturbances) becomes a subsonic pressure field with exponential decay upward (the positive y direction), and the incoming-wave family becomes a subsonic pressure field increasing exponentially upward.

Stability theory uses only the first solution. However, the second solution is the flow field produced by a moving wavy wall located above the boundary layer, and, therefore, in the present problem it represents the given external pressure field. When two independent numerical solutions are formed by integrating the inviscid equations through the boundary layer, the boundary condition of zero normal velocity ($v' = 0$) at the wall can be satisfied for any values of c_r and α (the dimensionless wave number). The situation in the viscous theory is similar, except that in addition to the incoming and reflected waves there are three viscous solutions which are also needed in order to satisfy the wall boundary conditions of zero velocity and temperature fluctuations.

The two computer programs have been modified to calculate the boundary-layer response to an incoming wave. In the inviscid theory, the two first-order equations are integrated for specified α and c_r for both an incoming wave of specified magnitude and a reflected wave of arbitrary magnitude. Both α and c_r are real, so that the waves are neutral in the sense of stability theory, and neither amplify nor damp with respect to space or time. What is being calculated is the steady-state response to a forcing function (the external pressure field) considered to have existed for an indefinite period of time. The amplitude and phase of the reflected wave are adjusted to satisfy the boundary condition at the wall, and then the equations are integrated again from the free stream to the wall for the combined solution to provide the final output. Because c_r is real, there is a singularity on the real y axis at the critical point ($U = c_r$). In the viscous theory, the eight first-order equations are integrated from the free stream to the wall for the three viscous solutions which go to zero as $y \to \infty$ along with the incoming and reflected wave solutions.

A combination of the five solutions satisfies the four boundary conditions at $y = 0$ for specified c_r, α, R (Reynolds number), and a specified magnitude of the incoming wave. In both theories about 50 quantities are computed at each output point. Chief among these are the rms fluctuation amplitudes and phase angles of the six flow variables (u', v', w', p', ρ', T' in conventional notation), plus the mass-flow and stagnation-temperature fluctuation amplitudes and phases, vorticity, Reynolds stress and energy-balance terms.

3. Numerical Results

A series of calculations have been carried out at $M_1 = 4.5$ with the two programs. This Mach number was chosen because it is the Mach number at which Kendall has made extensive measurements (SPS 37-62, Vol. III, pp. 43–47). The phase velocity used was $c_r = 0.65$, which is indicated by the measurements to be the most probable velocity of the sound sources in the turbulent side-wall boundary layers. The relative free-stream Mach number is then 1.575 for a two-dimensional wave ($\psi = 0$ deg). For oblique waves (in the plane formed by the u' and w' components), the relevant Mach number is that normal to the wave front. Hence for 0 deg $< \psi <$ 51 deg all disturbances are supersonic; for $\psi \geq 51$ deg they are subsonic. The phase velocity of the sonic two-dimensional disturbance is 0.778. Only supersonic disturbances will be considered here.

The dimensionless frequency used in stability theory is

$$F = \frac{\omega^* \nu_1}{U_1^{*2}} = \frac{\alpha c_r}{R}$$

The quantity ω^* is the circular frequency, and ν_1^* the free-stream kinematic viscosity coefficient. The reference length is $(x^* \nu_1^*/U_1^*)^{1/2}$, so that $R = R_x^{1/2}$, where R_x is the usual free-stream x-Reynolds number. Since c_r is constant, α/R is also constant for a fixed-frequency disturbance. The calculations were made for five frequencies in a range suggested by experiment. Since a hot-wire anemometer measures primarily local mass-flow fluctuations, attention will be directed to the ratio of the rms mass-flow fluctuation, $(MF)'$, to the mass-flow fluctuation of the incoming wave in the free stream at $y = 0$, $(MF)_I'$.

One of the curves in Fig. 1 shows the distribution through the boundary layer of $(MF)'/(MF)_I'$ for a typical case as computed from the viscous theory. There is a sharp peak near $y/\delta = 0.84$, and even though there is no amplification in the sense of stability theory, the peak $(MF)'$ in the boundary layer is 16 times the $(MF)'$ of the incoming wave alone. It can also be seen from Fig. 1 that in the free stream the $(MF)'$ of the combined incoming and reflected waves is less than 2. Also of interest is the fact that the peak $(MF)'$ occurs not near y_{cr}, the

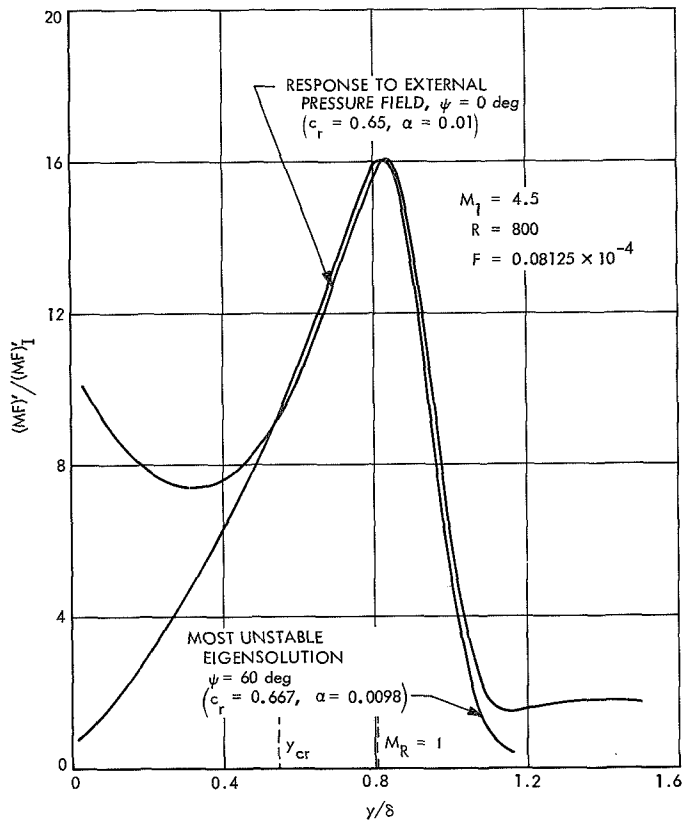
RESPONSE TO EXTERNAL PRESSURE FIELD, ψ = 0 deg (c_r = 0.65, α = 0.01)
M_1 = 4.5
R = 800
F = 0.08125 × 10^-4
MOST UNSTABLE EIGENSOLUTION
ψ = 60 deg
(c_r = 0.667, α = 0.0098)
y_cr
M_R = 1
(MF)'/(MF)'_1
y/δ

Fig. 1. Distribution through boundary layer of mass flow fluctuation amplitude

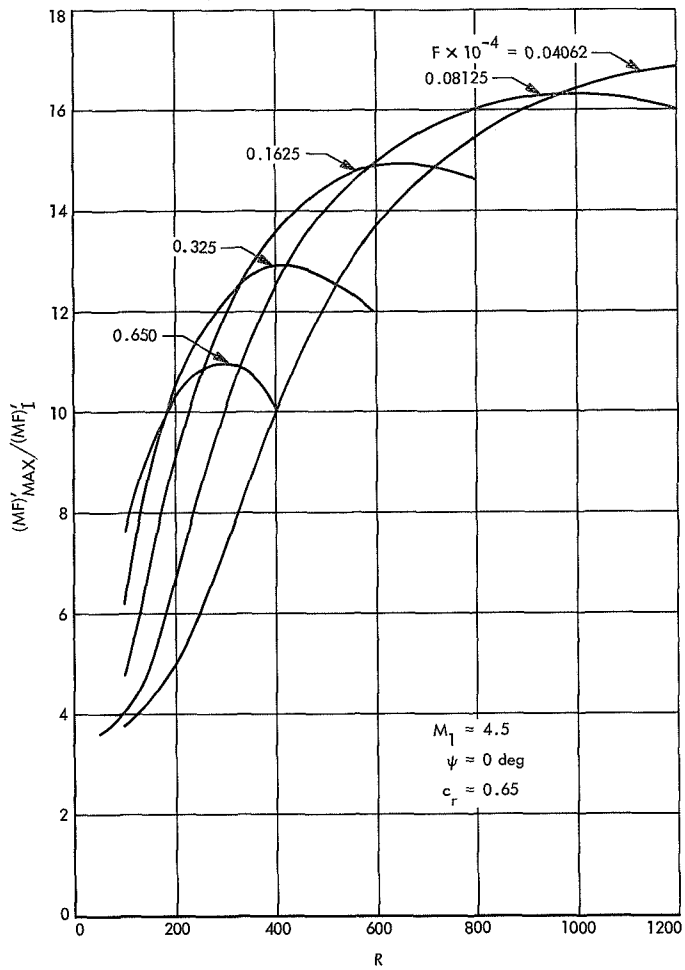

Fig. 2. Effect of frequency on boundary layer response to external pressure field

critical point, but near the point where the local relative Mach number is unity. It is not known whether this last feature is significant as the position of the peak decreases from $y/\delta = 0.96$ at $R = 50$ to 0.79 at $R = 1200$ for the frequency of Fig. 1. However, in the inviscid theory there seems to always be a peak near $M_R = 1$, except when masked by the singularity. The second curve in Fig. 1 shows $(MF)'$, scaled to the same peak value as the first curve, of the $\psi = 60$ deg eigensolution at the same Reynolds number and frequency. This eigensolution is slightly amplified with a temporal amplification rate about one fourth the amplification rate of the most unstable first-mode frequency at $R = 800$ ($F \times 10^4 \approx 0.6$). This particular comparison is given because it is desirable to know if the forced response of the boundary layer can be distinguished from instability waves, and since the $\psi = 60$ deg disturbances are the most unstable, they are the ones expected to occur in the boundary layer.

Figure 2 gives the ratio $(MF)'_{max}/(MF)'_1$, where $(MF)'_{max}$ is the peak value of $(MF)'$ in the boundary layer, as a function of R for five different frequencies with $\psi = 0$ deg. It is seen that $(MF)'_{max}$ increases steadily from the leading edge up to a certain Reynolds number whose value decreases with increasing frequency. Again it is emphasized that the disturbances are all neutral, but nevertheless there is an effective amplification of 10 to 20 times over the free-stream $(MF)'$ of the external pressure field. Once the disturbances reach the unstable region, they may be further amplified by the instability mechanism.

Figure 3 shows the effect of wave angle on the response for one of the frequencies of Fig. 2. The maximum response is found at $\psi = 45$ deg, and then falls off rapidly for higher angles. At $\psi = 50$ deg, the relative Mach number normal to the wave front is 1.012.

Finally, Fig. 4 shows the relation between the viscous and inviscid theories. The inviscid curve is not precisely located because the output points were spaced 0.07

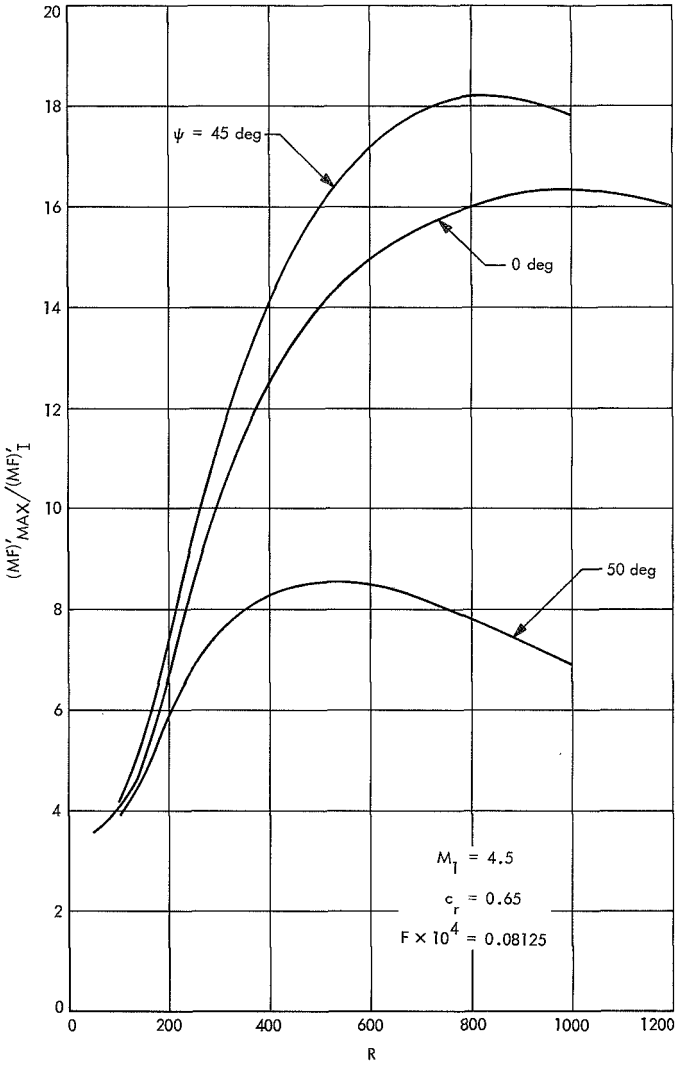

Fig. 3. Effect of wave angle on boundary-layer response to external pressure field

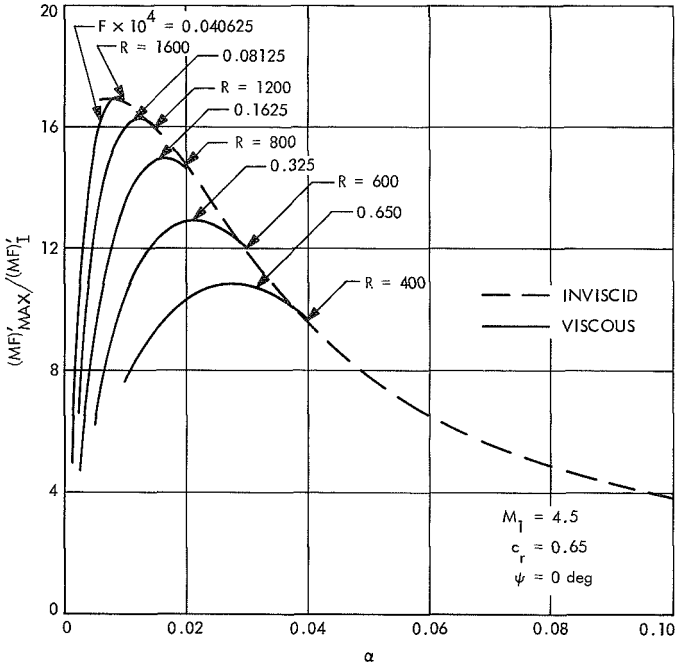

Fig. 4. Comparison of inviscid and viscous peak mass-flow fluctuations as functions of α

apart in y/δ, and the $(MF)'_{max}$ plotted is the largest numerical value occurring in the output. At sufficiently large α, the viscous and inviscid theories give almost identical results, except for the singularity and different boundary conditions of the inviscid theory. Since α increases proportionally to R for a fixed frequency, Fig. 4 shows that in the inviscid theory there is no apparent amplification at all. The peak mass-flow fluctuation decreases monotonically starting at the leading edge. The amplification region is a strictly viscous phenomenon. The viscous curves join the inviscid curve at an αR of about 18, so the longest viscous region occurs at the lowest frequencies.

Reference

1. Lees, L., and Lin, C. C., *Investigation of the Stability of the Laminar Boundary Layer in a Compressible Fluid,* NACA TN No. 1115. National Advisory Committee on Aeronautics, Washington, D.C., 1946.

B. Solution of Partial Differential Systems,

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Exact solutions of the sets of coupled nonlinear partial differential equations of physics (fluid physics, general relativity, etc.) are of great value for the physical insight they can give into extreme cases not susceptible to numerical or approximate treatment. The search for such solutions is, however, not a systematic process, depending in practice on intuitive skill, and physical or geometrical analogy for the divination of successful ansätze. In a first attempt at a more systematic methodology, we have considered, as a practical process of applied mathematics, the use of the generators of the invariance group of a given set of equations. Our method for finding these appears to be new; it is based on Cartan's geometric formulation of partial differential equations in the language of exterior differential forms, and introduces the fields of isovectors which generate the invariance transformations. With these we indeed find certain specializations of the set considered, which may be soluble exactly; in particular, we refer to classes of so-called similarity solutions and to algebraically special solutions based on invariant subspaces. Other methods for systematic discovery of exact solutions must also have close connections to this underlying group structure.

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A feature of this theory is that the operations for finding the isovector fields are: (1) algebraically routine and (2) involve many simultaneous first-order linear equations. As a consequence, formula manipulation by computer can have a very important application here. Having already achieved the capability of computer manipulation of exterior differential forms a FORMAC program, denoted ISOVECT, was developed which derives the equations for the generators of the invariance group of a given set of differential forms. It may ultimately prove feasible analytically to solve these equations by computer, also.

A detailed exposition of this new theory is to be published in the literature (Ref. 1). Application is there made to the one-dimensional heat equation (general similarity solutions are discussed), the vacuum Maxwell equations (algebraically special solutions), the Korteweg-de Vries equation of plasma physics (similarity solutions, conservation theorems), one-dimensional compressible fluid dynamics (hodograph solutions), the Lambropoulos equation (similarity solutions), and the cylindrically symmetric Einstein-Maxwell equations of general relativity (solution generation methods).

Reference

1. Harrison, B. K., and Estabrook, F. B., "Geometric Approach to Invariance Groups and Solution of Partial Differential Systems," *J. Math. Phys.* Vol. 12, Feb. 1971 (in press).